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New identities for the spectrum of the quantum Euler top

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Abstract
In this paper, we derive new identities for the spectrum of the quantum Euler top in terms of the sites $\alpha_0, \ldots, \alpha_n$ and the zeros of the joint eigenfunctions. Our identities improve previous formulas obtained by Kalnins and Miller (1992 *J. Phys. A: Math. Gen.* **25** 5663–75).

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1. Introduction

For any given set of positive parameters $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, the quantum Euler top is defined as the family of $n + 1$ partial differential operators

$$P_l = \sum_{i<j} \sigma_{ij}^{(l)}(\alpha) X_{ij}^2 \qquad (l = 0, \ldots, n).$$

Here $\sigma_{ij}^{(l)}$ denotes the lth symmetric function in the α parameters with α_i and α_j deleted, and X_{ij} is the killing vector field defined by rotation in the i, j plane, i.e.

$$X_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},$$

with $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$.

It is easy to verify that the P_l's are commuting, self-adjoint elliptic operators acting on the n-sphere S^n. Moreover, $P_0 = \Delta_{S^n}$, the constant curvature Laplacian on S^n. Consequently, the joint eigenfunctions are spherical harmonics that form a Hilbert basis of $L^2(S^n)$. The quantum Euler top constitutes an important example of quantum completely integrable systems [6, 9].

To describe the joint eigenfunctions in more details, it is customary to introduce elliptic or sphero-conal coordinates $(u_1, \ldots, u_n)$ on S^n. These are defined as the n zeros of the rational function

$$\sum_{j=0}^{n} \frac{x_j^2}{z - \alpha_j} = \frac{\prod_{j=1}^{n}(z - u_j)}{\prod_{j=0}^{n}(z - \alpha_j)}. \qquad (1.1)$$

Under the additional assumption $\sum_{j=0}^{n} x_j^2 = 1$, the u_j's form an orthogonal system of coordinates on S^n obtained from the intersection of the unit n-sphere with a family of confocal cones. Moreover, they satisfy the following interlacing property:

$$\alpha_0 < u_1 < \alpha_1 < u_2 < \alpha_2 < \cdots < \alpha_{n-1} < u_n < \alpha_n.$$

For any multi-index $\gamma = (\gamma_0, \ldots, \gamma_n) \in \{0, 1\}^{n+1}$, we denote the joint eigenfunctions of the operators P_l that are spherical harmonics of degree k by Ψ_k^γ and their corresponding eigenvalues by $\lambda_{k,l}^\gamma$, i.e.

$$P_l \Psi_k^\gamma = \lambda_{k,l}^\gamma \Psi_k^\gamma \qquad (l = 0, \ldots, n),$$

When separating the variables using elliptic coordinates, the joint eigenfunctions are expressed as the product

$$\Psi_k^\gamma(u_1, \ldots, u_n) = \prod_{j=1}^{n} \psi_k^\gamma(u_j) := \prod_{j=1}^{n} \prod_{i=0}^{n} |u_j - \alpha_i|^{\gamma_i/2} \cdot \phi_m^\gamma(u_j), \tag{1.2}$$

where ϕ_m^γ is a polynomial of degree $m := (k - |\gamma|)/2$. In addition, the function ψ_k^γ is a solution of the *generalized Lamé equation:*

$$\prod_{i=0}^{n}(x - \alpha_i) \frac{\mathrm{d}^2}{\mathrm{d}x^2} \psi_k^\gamma(x) + \frac{1}{2} \sum_{i=0}^{n} \prod_{k \neq i} (x - \alpha_k) \frac{\mathrm{d}}{\mathrm{d}x} \psi_k^\gamma(x) = \frac{1}{4} \left(\sum_{i=0}^{n-1} \lambda_{k,i}^\gamma x^{n-i-1} \right) \psi_k^\gamma(x). \tag{1.3}$$

For these reasons, the joint eigenfunctions Ψ_k^γ are often called *Lamé harmonics of degree k*. The corresponding function ψ_k^γ is called a *Lamé function of the first kind*, whereas ϕ_m^γ is called a *Lamé polynomial*. The polynomial

$$V_k^\gamma(x) = \sum_{i=0}^{n-1} \lambda_{k,i}^\gamma x^{n-i-1}$$

is often referred to as a *Van Vleck polynomial*. Note that the coefficients of V_k^γ are given by the joint eigenvalues $\lambda_{k,l}^\gamma$ of the P_l. For a more detailed derivation of these facts, we refer the readers to [1, 12].

Based on these observations, Kalnins and Miller ([9], equation (2.12)) obtained complicated formulas in terms of differential operators for the computations of the $\lambda_{k,l}^\gamma$'s as functions of the α_k's and the zeros of the Lamé polynomials. The purpose of this paper is to give explicit expressions that are much simpler than those of Kalnins and Miller. This is the content of the following section. In the last section, we give some applications to the asymmetric top.

2. Main result

We denote by $\theta_{k,1}^\gamma, \ldots, \theta_{k,m}^\gamma$ the zeros of the Lamé polynomials ϕ_m^γ appearing on the right-hand side of (1.2). Our goal is to establish the existence of simple relations for the eigenvalues $\lambda_{k,0}^\gamma, \ldots, \lambda_{k,n-1}^\gamma$ as functions of the parameters $\alpha_0, \ldots, \alpha_n$ and $\theta_{k,1}^\gamma, \ldots, \theta_{k,m}^\gamma$.

Before stating our main result, we need to recall some basic facts about the generalized Lamé equation. First, the zeros of V_k^γ as well as those of ϕ_m^γ are simple and lie within the interval (α_0, α_n). Moreover, none of the α_j is a zero of ϕ_m^γ. However, it is possible for V_k^γ to have a zero at some α_j for $j = 1, \ldots, n-1$. These results and their proofs can be found in [11].

Let $n_1 \in \{0, \ldots, n-1\}$ be the number of zeros of V_k^γ at the α_i's. Since the generalized Lamé equation (1.3) is invariant under permutations of the parameters α_j, we can reorder them in such a way that $\alpha_0, \ldots, \alpha_{n_1}$ are not zero of V_k^γ, whereas $\alpha_{n_1+1}, \ldots, \alpha_n$ are.

Theorem 2.1. *For $i = 0, \dots, n-1$, we have the expressions*

$$\lambda^{\gamma}_{k,n-i-1} = (-1)^{n-i-1} \sum_{j=0}^{n_1} \sigma_j^{(n-i)}(\alpha) \left[\sum_{l=1}^{m} \frac{4\gamma_j + 2}{\alpha_j - \theta^{\gamma}_{k,l}} + \sum_{\substack{l=0 \\ l \neq j}}^{n} \frac{2(\gamma_l + \gamma_j)^2}{\alpha_j - \alpha_l} \right], \tag{2.1}$$

where $\sigma_j^{(n-i)}(\alpha)$ denotes the $(n-i)$th symmetric function in the α parameters with α_j deleted.

Because Ψ_k^{γ} is an eigenfunction of Δ_{S^n} corresponding to a spherical harmonics of degree k, it follows that $\lambda^{\gamma}_{k,0} = k(k+n-1)$. If we apply the above result to the special case $i = n-1$, we obtain the interesting identity

$$k(k+n-1) = \sum_{j=0}^{n_1} \sigma_j^{(1)}(\alpha) \left[\sum_{l=1}^{m} \frac{4\gamma_j + 2}{\alpha_j - \theta^{\gamma}_{k,l}} + \sum_{\substack{l=0 \\ l \neq j}}^{n} \frac{2(\gamma_l + \gamma_j)^2}{\alpha_j - \alpha_l} \right], \tag{2.2}$$

where $\sigma_j^{(1)}(\alpha) = \alpha_0 + \dots + \alpha_{j-1} + \alpha_{j+1} + \dots + \alpha_n$.

Proof. For a matter of simplicity, we will drop the γ and k indices throughout the proof of theorem 2.1. The proof is a simple application of the residue calculus and the theory of Vandermonde matrices. We begin by dividing each side of the generalized Lamé equation (1.3) by $\prod_{j=0}^{n}(x-\alpha_j)\,\psi(x)$ to obtain

$$\frac{1}{4} \frac{\sum_{j=0}^{n-1} \lambda_{n-j-1} x^j}{\prod_{i=0}^{n}(x-\alpha_i)} = \frac{1}{\psi(x)} \left[\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{2} \left(\sum_{i=0}^{n} \frac{1}{x-\alpha_i} \right) \frac{\mathrm{d}\psi}{\mathrm{d}x} \right]. \tag{2.3}$$

The left-hand side of (2.3) has simple poles at $x = \alpha_0, \dots, \alpha_{n_1}$. Therefore, we can compute the residue at α_i on both sides of (2.3) to derive $n_1 + 1$ equations as follows. First, we have

$$\begin{aligned} \operatorname{Res}\left[\frac{\sum_{j=0}^{n-1} \lambda_{n-j-1} x^j}{\prod_{j=0}^{n}(x-\alpha_j)}, \alpha_i \right] &= \left[(x-\alpha_i) \frac{\sum_{j=0}^{n-1} \lambda_{n-j-1} x^j}{\prod_{j=0}^{n}(x-\alpha_j)} \right]_{x=\alpha_i} \\ &= \frac{\sum_{j=0}^{n-1} \lambda_{n-j-1} \alpha_i^j}{\prod_{j\neq i}(\alpha_i - \alpha_j)}. \end{aligned} \tag{2.4}$$

Second, elementary computations yield

$$\frac{\psi'(x)}{\psi(x)} = \sum_{j=1}^{m} \frac{1}{x-\theta_j} + \frac{1}{2} \sum_{i=0}^{n} \frac{\gamma_i}{x-\alpha_i}, \tag{2.5}$$

and

$$\begin{aligned} \frac{\psi''}{\psi} &= \sum_{i<j} \frac{2}{(x-\theta_i)(x-\theta_j)} + \left(\sum_{i=0}^{n} \frac{\gamma_i}{x-\alpha_i} \right) \left(\sum_{j=1}^{m} \frac{1}{x-\theta_j} \right) \\ &\quad + \frac{1}{4} \sum_{i=0}^{n} \frac{\gamma_i(\gamma_i - 2)}{(x-\alpha_i)^2} + \frac{1}{2} \sum_{i<j} \frac{\gamma_i \gamma_j}{(x-\alpha_i)(x-\alpha_j)}. \end{aligned} \tag{2.6}$$

Hence, (2.5) and (2.6) imply that

$$\begin{aligned}
&\mathrm{Res}\left[\frac{1}{\psi(x)}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\frac{1}{2}\left(\sum_{j=0}^{n}\frac{1}{x-\alpha_j}\right)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right),\alpha_i\right]\\
&\quad=\frac{1}{2}\sum_{j=1}^{m}\frac{1}{\alpha_i-\theta_j}+\frac{1}{4}\sum_{j\neq i}\frac{\gamma_i+\gamma_j}{\alpha_i-\alpha_j}+\sum_{j=1}^{m}\frac{\gamma_i}{\alpha_i-\theta_j}+\frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\alpha_i-\alpha_j}\\
&\quad=\frac{1}{2}\sum_{j=1}^{m}\frac{2\gamma_i+1}{\alpha_i-\theta_j}+\frac{1}{2}\sum_{j\neq i}\frac{(\gamma_i+\gamma_j)^2}{\alpha_i-\alpha_j}.
\end{aligned}\tag{2.7}$$

Combining equations (2.4) and (2.7), we conclude for $i=0,\ldots,n_1$ that

$$\frac{1}{2}\sum_{j=0}^{n-1}\lambda_{n-j-1}\alpha_i^j=\prod_{j\neq i}(\alpha_i-\alpha_j)\left(\sum_{j=1}^{m}\frac{2\gamma_i+1}{\alpha_i-\theta_j}+\sum_{j\neq i}\frac{(\gamma_i+\gamma_j)^2}{\alpha_i-\alpha_j}\right).\tag{2.8}$$

As for $\alpha_{n_1+1},\ldots,\alpha_n$, they must be zeros of the corresponding Van Vleck polynomial V, so

$$V(\alpha_i)=\sum_{j=0}^{n-1}\lambda_j\alpha_i^{n-1-j}=0\tag{2.9}$$

for $i=n_1+1,\ldots,n$.

Equations (2.9) and (2.8) can then be put in the matrix form $A\Lambda=2B$ with

$$A=\begin{pmatrix}1&\alpha_0&\alpha_0^2&\ldots&\alpha_0^n\\1&\alpha_1&\alpha_1^2&\ldots&\alpha_1^n\\\vdots&\vdots&\vdots&&\vdots\\1&\alpha_{n_1}&\alpha_{n_1}^2&\ldots&\alpha_{n_1}^n\\1&\alpha_{n_1+1}&\alpha_{n_1+1}^2&\ldots&\alpha_{n_1+1}^n\\\vdots&\vdots&\vdots&&\vdots\\1&\alpha_n&\alpha_n^2&\ldots&\alpha_n^n\end{pmatrix},\qquad\Lambda=\begin{pmatrix}\lambda_{n-1}\\\vdots\\\lambda_0\\0\end{pmatrix},$$

$$B=\begin{pmatrix}\prod_{j\neq0}(\alpha_j-\alpha_0)\left[\sum_{j=1}^{m}\frac{2\gamma_i+1}{\alpha_0-\theta_j}+\sum_{j\neq0}\frac{(\gamma_0+\gamma_i)^2}{\alpha_0-\alpha_j}\right]\\\vdots\\\prod_{j\neq n_1}(\alpha_j-\alpha_{n_1})\left[\sum_{j=1}^{m}\frac{2\gamma_i+1}{\alpha_{n_1}-\theta_j}+\sum_{j\neq n_1}\frac{(\gamma_{n_1}+\gamma_i)^2}{\alpha_{n_1}-\alpha_j}\right]\\0\\\vdots\\0\end{pmatrix}.$$

Since the α's are distinct, the matrix A is an invertible Vandermonde matrix. Using the results in [10], its inverse is given by

$$A^{-1}:=\left(\frac{(-1)^{i+1}\,\sigma_j^{(n-i)}(\alpha)}{\prod_{l\neq j}(\alpha_l-\alpha_j)}\right).\tag{2.10}$$

Consequently, the eigenvalues have the expressions

$$\lambda_{n-i-1} = (-1)^{n-i-1} \sum_{j=0}^{n_1} \sigma_j^{(n-i)}(\alpha) \left[\sum_{l=1}^{m} \frac{4\gamma_j + 2}{\alpha_j - \theta_l} + \sum_{\substack{l=0 \\ l\neq j}}^{n} \frac{2(\gamma_l + \gamma_j)^2}{\alpha_j - \alpha_l} \right] \tag{2.11}$$

as desired. □

Since the zeros of ϕ_m^γ represents the equilibrium positions of an electrostatic system with logarithmic potential [5], they must satisfy

$$\sum_{l=1}^{m} \frac{2\gamma_j + 1}{\theta_{k,l}^\gamma - \alpha_j} + \sum_{l\neq j} \frac{4}{\theta_{k,l}^\gamma - \theta_{k,j}^\gamma} = 0 \qquad (j = 0, \ldots, n). \tag{2.12}$$

These are usually known as the Niven equations [9]. From (2.12), we obtain the following identities.

Corollary 2.2. *For $i = 0, \ldots, n-1$, we have the expressions*

$$\lambda_{k,n-i-1}^\gamma = (-1)^{n-i-1} \sum_{j=0}^{n_1} \sigma_j^{(n-i)}(\alpha) \left[\sum_{l\neq j} \frac{8}{\theta_{k,j}^\gamma - \theta_{k,l}^\gamma} + \sum_{\substack{l=0 \\ l\neq j}}^{n} \frac{2(\gamma_l + \gamma_j)^2}{\alpha_j - \alpha_l} \right]. \tag{2.13}$$

In the special case $\gamma = 0$, the Lamé function ψ_k^γ reduces to the Lamé polynomial ϕ_k^γ. In that situation, our identities take the following simple form.

Corollary 2.3. *The joint eigenvalues of the operators P_l corresponding to eigenfunctions that are the product of Lamé polynomials ($\gamma = 0$) satisfy the identities*

$$\begin{aligned} \lambda_{k,n-i-1}^0 &= (-1)^{n-i-1} \sum_{j=0}^{n_1} \sum_{l=1}^{k/2} \frac{2\sigma_j^{(n-i)}(\alpha)}{\alpha_j - \theta_{k,l}^0} \\ &= (-1)^{n-i-1} \sum_{j=0}^{n_1} \sum_{l\neq j} \frac{8\sigma_j^{(n-i)}(\alpha)}{\theta_{k,j}^0 - \theta_{k,l}^0} \end{aligned}$$

for $i = 0, \ldots, n-1$.

3. Asymmetric top

In the important case $n = 2$, the quantum Euler top is better known as the quantum asymmetric top [1]. Namely, $P_0 = \Delta_{S^2}$ and

$$P_1 = L := \alpha_0 X_{12}^2 + \alpha_1 X_{02}^2 + \alpha_2 X_{01}^2.$$

From the introduction, we know that the joint eigenfunctions of Δ_{S^2} and L corresponding to spherical harmonics of degree k can be expressed as the product

$$\Psi_k^\gamma(u_1, u_2) = \psi_k^\gamma(u_1)\psi_k^\gamma(u_2),$$

where the function ψ_k^γ is given by

$$\psi_k^\gamma(x) = \prod_{j=1}^{3} |x - \alpha_j|^{\gamma_i/2} \phi_m^\gamma(x) \tag{3.1}$$

and $\phi_m^\gamma(x)$ is a polynomial of degree $m = (k - |\gamma|)/2$. In addition, $\psi_k^\gamma(x)$ is a solution of the *Lamé equation*:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_k^\gamma(x) + \frac{1}{2}\sum_{j=1}^{3}\frac{1}{x-\alpha_j}\frac{\mathrm{d}}{\mathrm{d}x}\phi_k^\gamma(x) = \frac{1}{4}\frac{k(k+1)x - \lambda_k^\gamma}{\prod_{j=1}^3(x-\alpha_j)}\phi_k^\gamma(x). \tag{3.2}$$

Note that λ_k^γ appearing on the right-hand side of (3.2) is the eigenvalue of L associated with the eigenfunction Ψ_k^γ. In order to simplify our formulas, we can use the fact that (3.2) is invariant under the linear transformation

$$T(x) = \frac{x-\alpha_0}{\alpha_2-\alpha_0}. \tag{3.3}$$

Under T, equation (3.2) is transformed into another Lamé equation with α's at 0, $a = \frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0} \in (0, 1)$, and 1. In that particular situation, theorem 2.1 yields

$$\tilde{\lambda}_k^\gamma = 2(\gamma_0+\gamma_1)^2 + 2a(\gamma_0+\gamma_2)^2 + 2a\sum_{j=1}^{m}\frac{2\gamma_0+1}{\tilde{\theta}_{k,j}^\gamma}, \tag{3.4}$$

where $\tilde{\lambda}_k^\gamma$ and $\tilde{\theta}_{k,j}^\gamma$ denote the eigenvalues and zeros when the α's are at $0, a, 1$. From the definition of T, it is easy to see that for general α's, the eigenvalues λ_k^γ and the zeros $\theta_{k,j}^\gamma$ are given by

$$\lambda_k^\gamma = \alpha_0 + (\alpha_2-\alpha_0)\tilde{\lambda}_k^\gamma \qquad \text{and} \qquad \theta_{k,j}^\gamma = \alpha_0 + (\alpha_2-\alpha_0)\tilde{\theta}_{k,j}^\gamma.$$

This yields the following result.

Theorem 3.1. *The eigenvalues of the asymmetric top L are either α_1 or satisfy the identities*

$$\lambda_k^\gamma = \alpha_0 + 2(\gamma_0+\gamma_1)^2 + 2(\alpha_1-\alpha_0)\left[(\gamma_0+\gamma_2)^2 + (\alpha_2-\alpha_0)\sum_{j=1}^{m}\frac{2\gamma_0+1}{\theta_{k,j}^\gamma-\alpha_0}\right].$$

As an interesting consequence of our last result, we can use the fact that $\lambda_k^\gamma \in (\alpha_0 k(k+1), \alpha_2 k(k+1))$ (see e.g. theorem 2.1 in [2]) to deduce

$$\sum_{j=1}^{m}\frac{1}{\theta_{k,j}^\gamma-\alpha_0} \leqslant \frac{\alpha_2}{\alpha_2-\alpha_0}k(k+1). \tag{3.5}$$

Similarly, we can use Niven's equation (2.12) to also obtain

$$\sum_{j=1}^{m}\frac{4}{\theta_{k,j}^\gamma-\theta_{k,0}^\gamma} \leqslant \frac{\alpha_2}{\alpha_2-\alpha_0}k(k+1). \tag{3.6}$$

From these inequalities, we deduce the following simple corollary.

Corollary 3.2. *Let $\theta_{k,1}^\gamma < \theta_{k,2}^\gamma$ denote the smallest two zeros of the Lamé polynomial ϕ_m^γ. We have*

$$\theta_{k,1}^\gamma - \alpha_0 \geqslant \frac{\alpha_2-\alpha_0}{\alpha_2 k(k+1)} \qquad \textit{and} \qquad \theta_{k,2}^\gamma - \theta_{k,1}^\gamma \geqslant \frac{4(\alpha_2-\alpha_0)}{\alpha_2 k(k+1)}.$$

4. Conclusion

Our methods remain valid in the case where we assume the parameters α to be distinct complex numbers. In that situation, the quantum Euler top has for complex analogs the Gaudin spin chains of various types [7, 8]. Consequently, our results can be extended to those cases.

Recently, there has been many papers on the asymptotic properties of the spectral data associated with the Euler top. However, many important questions are still open. One that is of great interest consists of computing the level spacings distribution (LSD) for the energy levels of the asymmetric top in the semi-classical regime ($k \to \infty$). Based on the Berry–Tabor conjecture in quantum chaos [4], one expects that the LSD for quantum completely integrable systems to follow a Poisson process. It would be interesting to see if the spectrum for the quantum asymmetric top also behaves like a sequence of independent random variables. In earlier work [3], we computed the LSD for the zeros of Lamé functions in various asymptotic regimes. We are hoping to combine these results with those presented in this paper to compute the LSD for the spectrum of the quantum asymmetric top in future research work.

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